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Chimeras

by

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The propagation of nonlinear wave packets and modulated beams, and other features of nonlinear wave propagation can be described in terms of several approximate, heuristic theories. The neoclassical approximation, to be presented here, is representative of a style of approximation that includes a number of the approximations currently in use as special cases or as limiting cases. It is, in a manner of speaking, the best approximation of its kind in the sense that it coincides with the nonlinear wave equation from which it is derived in more limiting cases than any of the others, and though it too is heuristic, it is rational, being the first of a sequence of approximations that converges to a solution if it converges. Chimeras are the solutions of the equations of the neoclassical approximation; there are a number of exact ones, to be exhibited here, that describe phase-shocks, self-focus~~s~~ed beams, and localized wave packets that travel, without change of shape, at an arbitrary uniform velocity less than one.

1. The body of a goat.

The search for exact solutions of nonlinear partial differential equations is a difficult game, and wins are not often seen unless you allow the player to choose his problems. This choosing of problems goes on at many levels. In physics, for example, and even in the most hopelessly specialized branches of physics, it is pretty well recognized that the notion of providing an exact theory that encompases the complete understanding of some class of phenomena is the perihelion of vanity. Instead, it is hoped that a model can be constructed - a model based on relatively simple principles that by logical inference, numerical or otherwise, predicts a body of observed results with a satisfactory degree of accuracy. It can hardly be thought surprising that a good deal of modern applied mathematics, with its emphasis on norms, contractions, invariant manifolds, and asymptotics, should be devoted to the determination of how satisfactory a given approximation really is and how different approximations are related to one another. Neither should it be thought surprising that simplicity of a physical theory is required, regardless of the complexity of the manipulations, if very many of us are to agree that we have achieved a deep understanding of some phenomenon.

Generally speaking, the choice of a problem or a model is influenced both by the nature of the phenomena to be described and by the mathematical techniques that are known or can be invented to deal with it. That many physical phenomena appear

to be described by nonlinear wave equations need not be argued, and if it is agreed therefore that it makes sense to look for new techniques to describe properties of solutions of nonlinear wave equations, then no more elaborate apology need be offered for considering

$$\phi_{tt} - \nabla^2\phi + \phi^3 = 0 \tag{1.1}$$

as an illustrative example. This equation (the $(\phi^4)_4$ theory) is one of the most popular equations of theoretical physics - it packs Lorentz invariance, nonlinearity of the right kind for it to be a wave equation, phase velocities greater than one and group velocities less than one, and reduction to a non-dispersive equation in the weak field limit, all in less than a square inch of the printed page. Its simplicity is evidently measured by the small number of symbols it takes to write it down. There is one more simplifying feature of (1.1) worthy of note - it is derivable from a stationary principle. In the notation to be used throughout this paper, it is the Euler equation of

$$\delta\int\{\tfrac{1}{2}\, g^{ij}\phi_{,i}\phi_{,j} - \tfrac{1}{4}\,\phi^4\}\, d^4x = 0 , \tag{1.2}$$

where $g_{ij} = g^{ij} = 1$ if $i=j=0$, $=-1$ if $i=j\neq 0$, and $=0$ otherwise, a comma followed by one or more subscripts denotes differentiation with respect to the corresponding coordinates $(x^k) = (t,\underline{x})$ $(k=0,1,2$ and $3)$, and the summation over repeated indices is implied.

If simplicity of a theory that encompases specified properties such as those listed above of the $(\phi^4)_4$ theory is to be measured by the number of symbols required to write it, then (1.1) is undoubtedly the simplest theory of its kind. On the other hand if theories are thought to become simpler as the number of their easily found exact solutions that mimic physical phenomena increases, then

$$(1.3) \qquad \begin{aligned} & g^{ij}(a^2P_{,i})_{,j} = 0 \\ & g^{ij}a_{,ij} = (g^{ij}P_{,i}P_{,j}-a^2)a \end{aligned}$$

is a far simpler theory of the same kind. As we shall see in the next section (1.3) is also very closely related to (1.2).

2. A serpent's tail.

By the counting of symbols

$$\delta\int\mathcal{L}(\phi,\phi_{,i})d^4x = 0 \tag{2.1}$$

is simpler, though considerably more general than (1.2), and it is usually quite a bit more complicated. A number of the properties of (1.2) can be stated in terms of it, however, and that will be done wherever it is easy. Nevertheless, this section is devoted primarily to an exploration of the relation of (1.3) to (1.2).

There is one well-known class of exact solutions of (1.2): the plane waves defined by the ansatz $\phi = f(k_i x^i)$ where k_i is a constant four-vector. The resulting ordinary differential equation for f has the energy integral

$$\frac{1}{2}\,k^i k_i f'^2 + \frac{1}{4}\,f^4 = E \tag{2.2}$$

where the prime denotes differentiation of f with respect to its argument. From (2.2) we can determine f by a quadrature, evaluate the action

$$J(k^i k_i, E) \equiv \oint (2k^i k_i (E - \tfrac{1}{4}\,f^4))^{\frac{1}{2}} df = 2\pi\gamma(k^i k_i)^{\frac{1}{2}}\,E^{3/4} \tag{2.3}$$

where $$\gamma = (\tfrac{1}{2})!\,(-\tfrac{3}{4})!/\pi(\tfrac{3}{4})! \approx 1.1128\ ,$$

and define the dispersion relation

$$J_E = 2\pi \qquad \underline{\text{or}} \qquad k^i k_i = (\tfrac{4}{3\gamma})^2\,E^{\frac{1}{2}}\ . \tag{2.4}$$

The condition (2.4) that the period of f shall be 2π is required if the components k^i are to be the frequency and the wave number vector, by the usual convention for plane waves. Similar features of (2.1) can be isolated whenever the same ansatz leads to an ordinary differential equation that has periodic solutions, and for present purposes a sufficiently sharp definition of a nonlinear (or linear) wave equation is: one that has plane wave solutions.

Now let us suppose that (2.1) is a nonlinear wave equation. Then we may ask if there is a class of approximate solutions that are close to plane waves - close in a sense not to be made precise just yet, for it is a difference in the notions of closeness that distinguishes the classical approximation from the neoclassical approximation. Both terms will be defined presently. In either approximation we wish to represent the function $\phi(x^k)$ which is to be a solution of (2.1) in such manner as to display explicitly a phase $P(x^k)$ that takes on the values $k_i x^i$ if the solution is a plane wave. The familiar phase-amplitude representation, $\phi(x^k) = A(x^k)\sin P(x^k)$, though perfectly faithful (any ϕ can be so represented), leads when substituted in (2.1) to equations that are not particularly illuminating unless the Euler equation of (2.1) happens to be linear. Later we shall return to the phase-amplitude representation (slightly modified) with the equations of the neoclassical approximation to govern it, but first let us engage in a bit of circumlocution based on the more general representation

$$(2.5) \qquad \phi(x^k) = f(\theta, x^k) \quad \text{when} \quad \theta = P(x^k)$$

where f is a 2π-periodic function of its first argument.

<u>Representation theorems</u>.

Equation (2.5), by the way it was written, was meant to imply that we shall view the functions $\phi(x^k)$ (of four variables) as projections of the functions $f(\theta, x^k)$ (of five variables) defined by the substitution of $P(x^k)$ for θ. The question to be answered by the first representation theorem is whether or not there is a partial differential equation in the five independent variables whose solutions when projected are solutions of (2.1). The hard part of the theorem is to know it is needed - its form is easy to guess.

Theorem 1: Given any C^2 function $P(x^k)$, every function $f(\theta, x^k)$ that is a solution of

$$(2.6) \qquad \delta \int \mathscr{L}(f, P_{,i} f_\theta + f_{,i})\, d^4x\, d\theta = 0$$

is projected onto a solution of (2.1) by the substitution $\theta = P(x^k)$.

The proof of theorem 1 is embarrassingly easy - the substitution of $f(P(x^k), x^k)$ for $\phi(x^k)$ in the Euler equation of (2.1), the adoption of the convention that the subscript θ denotes differentiation of f with respect to its first argument, the use of the chain rule for partial differentiation, and the observations

$\mathscr{L}_f = \mathscr{L}_\phi$, $\mathscr{L}_{f_{,i}} = \mathscr{L}_{\phi_{,i}}$ and $\mathscr{L}_{f_\theta} = P_{,i}\mathscr{L}_{f_{,i}}$, all conspire to produce the Euler equation of (2.6). Moreover, the proof is considerably more general than the theorem. The function f needn't be a 2π-periodic function of θ (that and the corresponding choice of $(0,2\pi)$ for the domain of the θ-integration in (2.6) are side conditions that identify $P(x^k)$ as the phase of a wave), the equation for ϕ needn't be an Euler equation, ϕ and f needn't be scalars, and the number of independent phases that can be introduced is not limited to one. In short, every feature of the theorem can be generalized, but we shall find little use for such generalizations here.

Theorem 1, by nature a statement about projections, defines ϕ uniquely given P and f, and the converse, that f should be uniquely determined given P and ϕ, may or may not be true depending on the choice of P. There is always the <u>trivial</u> representation where $\phi(x^k) = f(\theta,x^k)$ and nothing is said about evaluating f at $\theta = P(x^k)$. Then f is independent of θ , f_θ and with it $P_{,i}$ are expunged from (2.6), and (2.1) is recovered directly. What we should like to find is a condition, an extra equation that can be thought of as governing the choice of P, that allows us to consider nontrivial representations where f_θ does not vanish identically - in the argot of applied mathematicians, a bifurcation condition. Among other things, the second representation theorem provides it. Let us suppose that we have a solution $f(\theta,x^k)$ and that it can be represented as

$$(2.7) \qquad f(\theta,x^k) = F(\theta;\{A_n(x^k)\})$$

where the θ-dependence of F is specified and $\{A_n\}$ denotes an indefinite but countable number of parameters. The Fourier representation of f where the θ-dependence of F is prescribed in terms of the trigonometric functions is the most familiar one of the form (2.7), but for reasons shortly to be seen we shall not limit (2.7) to that special case. Given $F(\theta;\{A_n\})$ we can evaluate

$$(2.8) \qquad \mathcal{P}(\{A_n\},\{A_{n,i}\},P_{,i}) \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}(F,P_{,i}F_\theta+\Sigma A_{n,i}F_{A_n})d\theta \ ,$$

and in terms of it, state the second representation theorem.

Theorem 2: If $f(\theta,x^k)$ is in the class of functions defined by (2.7) and is a solution of (2.6), then

$$(2.9) \qquad \delta\int \mathcal{P}(\{A_n\},\{A_{n,i}\},P_i)d^4x = 0 \ .$$

The proof of theorem 2 is by a direct evaluation of the Euler equation associated with the n^{th} component of the parametric dependence of F,

$$(2.10a) \qquad (\mathcal{P}_{A_{n,i}})_{,i} - \mathcal{P}_{A_n} = \frac{1}{2\pi}\int_0^{2\pi} \{(F_{A_n}\mathcal{L}_{F_{,i}})_{,i} - F_{A_n\theta}\mathcal{L}_{F_\theta} - F_{A_n,i}\mathcal{L}_{F_{,i}}$$

$$- F_{A_n}\mathcal{L}_F\}d\theta = \frac{1}{2\pi}\int_0^{2\pi} F_{A_n}\{(\mathcal{L}_{F_\theta})_\theta + (\mathcal{L}_{F_{,i}})_{,i} - \mathcal{L}_F\}d\theta = 0 \ ,$$

and by the observation,

$$(2.10b)\quad \frac{1}{2\pi}\mathcal{L}\Big|_0^{2\pi} = \frac{1}{2\pi}\int_0^{2\pi}\{F_\theta\mathcal{L}_F + F_{\theta\theta}\mathcal{L}_{F_\theta} + F_{\theta,i}\mathcal{L}_{F,i}\}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\{F_\theta[\mathcal{L}_F - (\mathcal{L}_{F_\theta})_\theta - (\mathcal{L}_{F_{,i}})_{,i}] + (F_\theta\mathcal{L}_{F_{,i}})_{,i}\}d\theta$$

$$= (\overline{\mathcal{L}}_{P_{,i}})_{,i} = 0 \,.$$

That F should actually depend on θ and $\{A_n\}$ is implicit in (2.7), and (2.10b) is evidently the necessary condition for bifurcation of the representation (2.5). Now let us see how (2.6) and (2.9) which appear to define harder problems than (2.1) can be used to derive the classical and neoclassical approximations.

<u>The classical approximation</u> - <u>perturbation theory</u>.

A great deal of work has already been done on the approximate solution of wave propagation problems in a limit where the solution can be thought of as a local oscillation described parametrically by slowly varying functions of space and time - the WKBJ approximation, geometrical optics, Whitham's method, and a host of other coefficient averaging techniques are such perturbation theoretic approximations. In terms of the representation (2.5) a precise definition of <u>slowly varying</u> can be stated succinctly - the perturbation theories are limited to the treatment of solutions where

$$(2.11)\qquad \|f_{,i}\| \le \varepsilon\|P_{,i}f_\theta\|,\ 0 \le \varepsilon \ll 1,$$

and $\|\cdot\|$ denotes the maximum of the absolute value over the range of values of θ. The easiest way to arrive at a formal asymptotic perturbation theory now is to introduce the rescaling of the independent variables

$$X^k = \varepsilon x^k \tag{2.12}$$

and in accord with (2.12), the representations

$$\phi(x^k) = f(\theta, X^k, \varepsilon) = F(\theta; \{A_n(X^k, \varepsilon)\}) \text{ when } \theta = \frac{P(X^k, \varepsilon)}{\varepsilon} \tag{2.13}$$

where f and F are 2π-periodic functions of θ. Then

$$\begin{aligned} &\mathcal{L}(\phi, \phi_{,i}) = \mathcal{L}(f, P_{,i} f_\theta + \varepsilon f_{,i}) \text{ and} \\ &\mathcal{G} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}(F, P_{,i} F_\theta + \varepsilon \Sigma A_{n,i} F_{A_n}) d\theta \end{aligned} \tag{2.14}$$

where the comma followed by a subscript now denotes differentiation with respect to x^i or X^i, depending on which arguments are listed in (2.13). In the rescaled variables ε no longer appears on the right hand side of the equation that corresponds to (2.11), and the slowly varying property that was assumed of solutions there reappears in the form of an explicit dependence of the Euler equations on the fictitious parameter ε. A few comments on how theorems 1 and 2 impinge upon and simplify the structure of the perturbation theory can now be made:

1) The rescaling (2.12) casts the problem in the form of a <u>two-scale</u> perturbation theory, a device that has achieved considerable popularity lately (cf. Cole [1]), and

theorems 1 and 2 provide the justification for doing so <u>in advance</u>.

2) Given a sequence $\{\phi^{(n)}(x^k)\}=\{f^{(n)}(P^{(n)}(x^k),x^k)\}$ for which $\|f^{(n+1)}-f^{(n)}\| = 0(\varepsilon\|f^{(n)}-f^{(n-1)}\|)$ and $|P^{(n+1)}-P^{(n)}|=0(\varepsilon|P^{(n)}-P^{(n-1)}|)$ in some region of space and time, it does <u>not</u> follow that $|\phi^{(n+1)}-\phi^{(n)}|=0(\varepsilon|\phi^{(n)}-\phi^{(n-1)}|)$ in the same region for the reason that the zeroes of the functions $\phi^{(n)}$ are not fixed, but vary slightly in accord with $P^{(n)}(x^k)$. The representation (2.5) allows the introduction of the norm defined after (2.11) in terms of which error bounds can be treated (cf. Bisshopp [2]).

3) In the direct treatment of (2.1) by a two-scale method (where (2.12) and (2.13) are substituted in the Euler equation, cf. Luke [3]) $P^{(n)}$ is determined at each stage of the approximation by an integrability condition for $f^{(n+1)}$. Theorem 2 replaces the sequence of integrability conditions by a single bifurcation condition without reference to asymptotics.

In practice the error bounds associated with a sequence of approximations obtained by successive substitutions in (2.14) have a more complicated form than has been exhibited in point 2 above. The norm of the error at the n^{th} stage depends on norms of the errors of the X^k-derivatives at the previous stage, indicating that it is a singular perturbation theory we have to deal with. Rather than to proceed further into the rafted pack of asymptotics, error bounds, and numerous expansions in powers of things, however, it will be sufficient for present purposes to outline the consequences of the observation that the

approximation of lowest order - the classical approximation - can be isolated by neglecting $f_{,i}$ relative to $P_{,i}f_\theta$ in (2.6) and $(\Sigma A_{n,i}F_{A_n})$ relative to $P_{,i}F_\theta$ in (2.8). Accordingly, we shall disregard the rescaling of variables (2.12) from here on and suppress the superscript (0) in the discussion of the classical approximation.

When $f_{,i}$ is removed from (2.6) the Euler equation

$$(2.15) \qquad (\mathcal{L}_{f_\theta})_\theta = \mathcal{L}_f$$

is an ordinary differential equation (given $P_{,i}$), and it has the energy integral

$$(2.16) \qquad H(f,g;P_{,i}) \equiv gf_\theta - \mathcal{L}(f,P_{,i}f_\theta) = E(x^k)$$

$$g \equiv \mathcal{L}_{f_\theta}$$

(provided the Legendre condition $g_{f_\theta} \neq 0$ is satisfied). From (2.16) the local waveform $f(\theta;P_{,i},E)$ can be obtained by a quadrature, but it is not necessary to perform that step to evaluate the action

$$(2.17) \qquad J(P_{,i},E) \equiv \oint_{H=E} g\,df$$

(Note that the local wave is an exact solution of (2.1) whenever $P_{,i}$ is a constant vector k_i and E is another constant.) Since $P_{,i}$ and E are the only parameters of the local waveform every representation of the form (2.7) has coefficients of the form $\{A_n(P_{,i},E)\}$: this and the observation that

$$(2.18)\qquad 2\pi\mathcal{L}(\{A_n\},P_{,i}) = \int_0^{2\pi} (gf_\theta - H)\,d\theta = J(P_{,i},E) - 2\pi E$$

when $F_{,i}$ is removed from (2.8) imply the relations

$$(2.19)\qquad (J_{P_{,i}})_{,i} = 0 \qquad\qquad \text{(transport equation)}$$

and

$$(2.20)\qquad J_E = 2\pi \qquad\qquad \text{(dispersion relation)}$$

that govern the classical approximation (cf. Whitham [4]). Unlike geometrical optics where the dispersion relation depends only on the frequency and the components of the wave number vector, equations (2.19) and (2.20) are coupled when (2.1) has a nonlinear Euler equation. Since J_{EE} is not zero in general the dispersion relation can be solved for $E(P_{,i})$ (in principle), and the result substituted in the transport equation to obtain the second order, quasilinear partial differential equation

$$(2.21)\qquad P_{,ij}(J_{EP_{,i}}J_{EP_{,j}} - J_{EE}J_{P_{,i}P_{,j}}) = 0$$

for $P(x^k)$. In the cases that have been encountered in practice so far (2.21) is not ultrahyperbolic; the qualitative behavior of its solutions has depended on whether it is elliptic or totally hyperbolic (cf. Courant-Hilbert [5]). If it is elliptic, initial value problems where $P(x^k)$ and $E(x^k)$ are prescribed in a three dimensional region of space-time can be posed without contradiction, but they are not well-posed, for the solutions are unstable - minute perturbations of the initial data grow faster than exponentially to become singularities of the perturbed solution.

(The spike instability of high intensity laser beams and the catastrophic self-focussing of laser beams in benzene are physical phenomena that appear to be represented by such behavior.) If (2.21) is totally hyperbolic, there are three-dimensional regions of space-time where the specification of $P(x^k)$ and $E(x^k)$ defines a well posed (stable) initial value problem, but then there is a ubiquitous tendency for the solutions to become multi-valued, or at least discontinuous if we presume that shocks (shadow boundaries) are formed. In either case it is predicted that the development of the solution is such as to bring about the violation of one or more of the inequalities (2.11) on which the classical approximation is squarely based. In spite of this, the classical approximation (when applied to (1.2) for example) has solutions whose qualitative features reflect many of the effects observed in laser beam experiments - it would be a shame to discard it just because it has suicidal tendencies. Perhaps with a little more analysis ...

The neoclassical approximation - direct methods.

Just as countless eigenvalue problems governed by stationary principles can be attacked by a direct method (the variational method), so too can (2.6). There are many ways to formulate a direct method - one of the easiest ones, though not the most convenient one for a nonlinear problem, is by harmonic analysis. Suppose we let

$$(2.22) \quad f(\theta,x^k) = F(\theta;\{A_n\}) = \sum_1^\infty A_n(x^k)\sin n\theta \quad .$$

Then, to state the weakest possible converse of theorem 2, if the Fourier series converges to a C^2 function when $\{A_n(x^k)\}$ and $P(x^k)$ are solutions of (2.9), then $f(\theta,x^k)$ is an odd, periodic solution of (2.6) and a nontrivial representation of a solution of (2.1). Of course the Euler equations of (2.9) are the infinite set of coupled nonlinear partial differential equations

(2.23) $$(\mathcal{L}_{P_{,i}})_{,i} = 0$$

and

$$(\mathcal{L}_{A_{n,i}})_{,i} = \mathcal{L}_{A_n} \quad \text{for} \quad n=1,2,\dots ,$$

and finding their exact solutions is out of the question. Because of the special properties of the trigonometric functions, however, (2.22) may be the most convenient form of (2.9) for proving existence of solutions and such things.

In the case where $P_{,i}$ is the constant vector k_i and $\{A_n\}$ is a set of constants we can be sure there is a solution of (2.23) - the plane wave - determined by equation (2.16). Because of the representation (2.22), however, (2.23) determines the Fourier series of the plane wave, and that simply is not the most convenient way to describe solutions of (2.16) in general. Furthermore, the one-term approximation, i.e. the phase-amplitude representation where $F(\theta;A) = A(x^k)\sin\theta$, is at best approximation, even for plane waves, when (2.1) is a nonlinear wave equation. If, however, the one-term Fourier approximation is employed in the analysis of (1.2) according to (2.23), the result is (1.3) with

$a = (3/4)^{\frac{1}{2}} A$;   it is the same result as would be obtained by applying the quasi-optical approximation to (1.2) (cf. Akhmanov et al [6]).

An improvement over the one-term Fourier approximation can be made if we are willing to discard (2.22) in favor of what might be called anharmonic analysis according to the scheme.

$$(2.24) \quad f(\theta,x^k) = F(\theta;\{A_n\},P_{,i}) = F_1(\theta;P_{,i}) + \sum_2^{\infty} A_n(x^k)F_n(\theta;P_{,i})$$

where F_1 is the fundamental 2π-periodic, odd solution of (2.16) (it depends parametrically on $P_{,i}$, but not on E since $J_E = 2\pi$ can be solved for $E(P_{,i})$ when $J_{EE} \neq 0$) and $\{F_n\}$ is a complete, orthogonal set of odd, 2π-periodic functions. Given F_1 the remaining members of the set can be constructed in many ways, and since the first member depends parametrically on $P_{,i}$ the others will too in general. Now (2.24) is a slight generalization of the representation (2.7), and instead of (2.8) we obtain

$$(2.25) \quad \mathcal{G}(\{A_n\},\{A_{n,i}\},P_{,i},P_{,ij}) \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}(F,P_{,i}F_\theta + P_{,ij}F_{P,j} + \Sigma A_{n,i}F_{A_n})d\theta \ ,$$

A reinvestigation of theorem 2 shows that its generalization is still true under the more general representation (2.25), for by a direct computation we obtain the Euler equations,

$$(2.26a) \qquad (\mathcal{G}_{A_{n,i}})_{,i} = \mathcal{G}_{A_n} \quad \text{for} \quad n = 2,3,\ldots$$

as before and

$$(2.26\text{b}) \qquad (\bar{\mathcal{G}}_{P,ij})_{,ij} = (\bar{\mathcal{G}}_{P,i})_{,i}$$

instead of (2.10b). If the classical approximation were to be redone here, the result would be (2.21), directly.

Let us define the neoclassical approximation now as the one-term approximation obtained by retaining only $F_1(\theta;P_{,i})$ in (2.24)*. It has the following properties:

(1) It is the first of a sequence of truncations of a direct method - if a sequence of solutions of the truncated approximations converges to a C^2 function, the limit is a solution of (2.6).

(2) It is exact for plane waves.

(3) Its classical approximation (the leading terms of the perturbation theory when (2.11) is assumed) is the classical approximation of the full problem (2.1).

(4) It shows clearly that the classical approximation is derived from a singular perturbation theory in which higher derivatives than those retained are neglected, and since it contains the higher derivatives *ab initio* it does not have the self-destructive tendencies of the classical approximation.

Let us put generalities aside now, and resume the discussion of (1.2). From (2.2) it follows that the plane wave has a universal waveform regardless of the value of k^ik_i. To see that this is so let $\theta = k_ix^i$ and $f(\theta) = (k^ik_i)^{\frac{1}{4}}F(\theta)$; then

*A similar approximation obtained by a perturbation theory has been reported by Chu and Mei [7].

$$\frac{1}{2} F'^2 + \frac{1}{4} F^4 = \frac{E}{(k^i k_i)^2} = \left(\frac{3\gamma}{4}\right)^4 \tag{2.27}$$

follows from (2.2) and (2.4). In this case the anharmonic analysis of (2.24) can be simplified to

$$f(\theta, x^k) = F(\theta; A, \{A_n\}) = A(x^k)F_1(\theta) + \sum_2^\infty A_n(x^k)F_n(\theta) \tag{2.28}$$

where F_1 is the solution of (2.27) (automatically 2π-periodic) and $\{F_n\}$ is complete, orthogonal, odd, and 2π-periodic. The neoclassical approximation is

$$g^{ij}(A^2 P_{,i})_{,j} = 0 \qquad \text{(transport equation)} \tag{2.29}$$

and

$$g^{ij}A_{,ij} = \beta g^{ij} P_{,i} P_{,j} A - \alpha A^3 \qquad \text{(diffraction-dispersion eq.)} \tag{2.30}$$

where

$$\beta = \int_0^{2\pi} F_1'^2 d\theta \Big/ \int_0^{2\pi} F_1^2 d\theta \quad \text{and} \quad \alpha = \int_0^{2\pi} F_1^4 d\theta \Big/ \int_0^{2\pi} F_1^2 d\theta \ .$$

Of course (2.29, 30) is not quite the same as (1.3), but by a closer examination of (2.27) it can be seen that F_1 cannot differ from $(4/3)^{\frac{1}{2}}$ sin θ by more than a few percent, and therefore $\beta \approx 1$ and $\alpha \approx 1$. If we set β=1 and $a=(\alpha)^{\frac{1}{2}}A$, (1.3) follows - for all practical purposes it is the neoclassical approximation of (1.2). Thus for equation (1.2) in particular it makes little difference whether we employ the one-term Fourier approximation or the neoclassical approximation.

3. And the head of a lion.

Some understanding of how the death-wish of the classical approximation is mollified in the neoclassical approximation can be gained by looking at some exact solutions. The class of solutions we shall discuss stands in relation to (1.3) in precisely the position of the plane waves relative to (1.2) - they are solutions that do not vary with time in a coordinate system that moves with a constant velocity relative to whatever coordinate system was implied in the writing of (1.2). The major differences are that the plane waves are periodic and they travel with phase velocities greater than one while the chimeras are not necessarily periodic and they travel with group velocities less than one. Since the Lorentz invariance of (1.2) is perfectly reflected in (1.3) a chimera can be described in its rest frame where $P_{,0}$ ($\equiv\omega$) is a constant (as implied by the integrability condition $P_{,0j} = P_{,j0} = 0$) and $a_{,0}$ is zero, then set in motion by a Lorentz transformation. In the rest frame of such a chimera

$$\underline{\nabla}\cdot(a^2\underline{\nabla}P) = 0 \tag{3.1}$$

and

$$\nabla^2 a+(\omega^2-|\underline{\nabla}P|^2-a^2)a = 0\ . \tag{3.2}$$

(Note that $\phi(x^k) \sim A(\underline{x})\ F(\omega t+P(x))$ is not time independent in any frame unless it is a plane wave.)

Plane chimeras

First let us consider the case where a and $\underline{\nabla}P$ depend on the single cartesian coordinate x. The integrability conditions $P_{xy} = P_{yx} = P_{xz} = P_{zx} = 0$ imply P_y and P_z are constant, the coordinates can be chosen so that P_z is zero, and the value k assigned to P_y provided $k^2 < \omega^2$. Then

$$(3.3) \qquad P_x = \frac{\ell}{a^2}\,, \quad a_{xx} + (\omega^2 - k^2 - \frac{\ell^2}{a^4} - a^2)a = 0$$

where ℓ is a constant. Equation (3.3) has the energy integral

$$(3.4) \qquad \frac{1}{2}\,a_x^2 + V(a) = \varepsilon\,, \quad V(a) = \frac{1}{2}(\omega^2 - k^2)a^2 + \frac{1}{2}\,\frac{\ell^2}{a^2} - \frac{1}{4}\,a^4\,,$$

and for $\ell \neq 0$ the qualitative structure of the solutions can be inferred from figure 1.

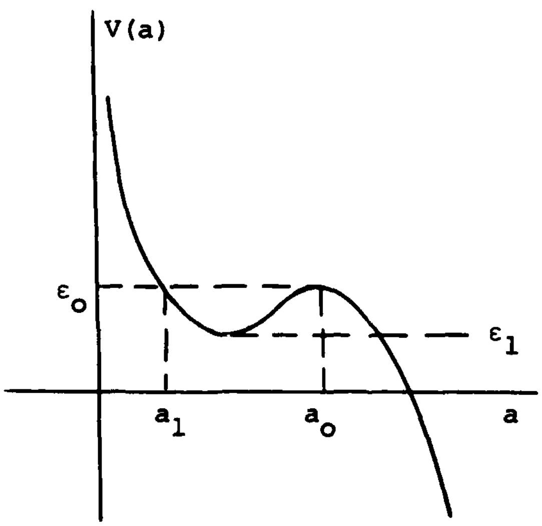
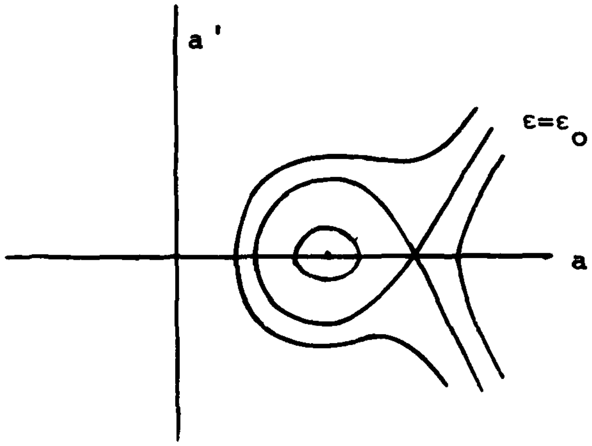
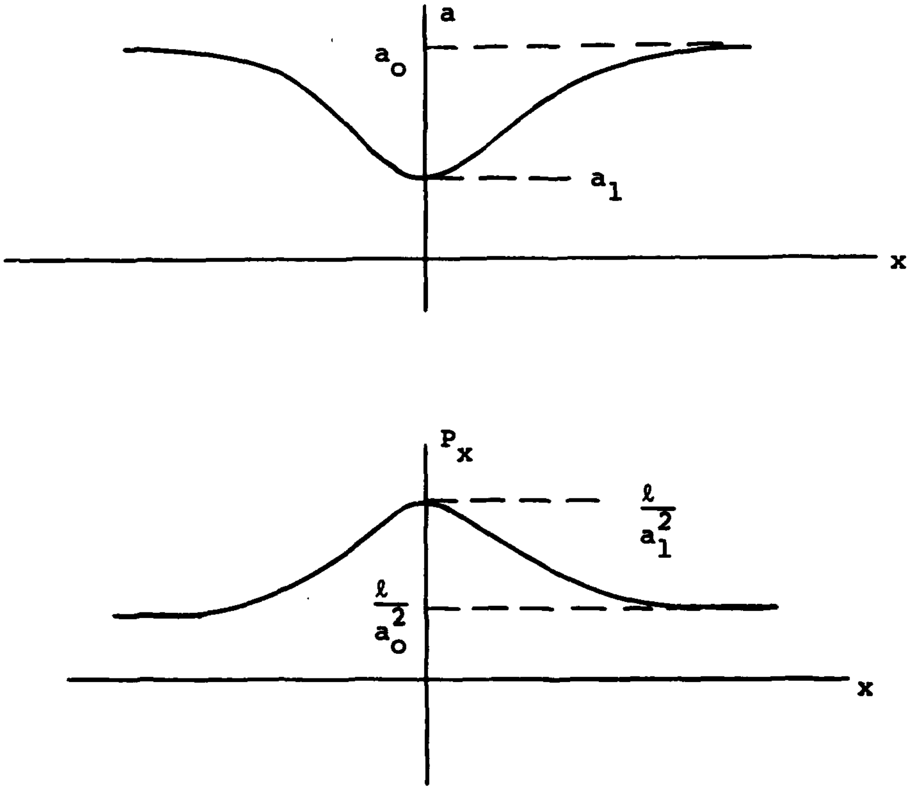


Figure 1a. V(a) .



<u>Figure 1b</u>. The phase plane.



<u>Figure 1c</u>. ($\varepsilon=\varepsilon_o$) Solutions.

Of course there are periodic plane chimeras, more or less like the cnoidal waves of fluid dynamics, when ε lies between ε_1 and ε_0, but the most interesting one is the one depicted in figure 1c where $\varepsilon = \varepsilon_0$. By the exquisite balance of diffraction (as represented by a_{xx}) and nonlinear dispersion (as represented by everything else) the amplitude of a plane wave dips from the value a_0 to a value a_1 and returns across a line inclined at some angle to the direction of its propagation while the value of P_x increases to an arbitrarily high multiple of its original value (if ℓ is sufficiently small) and returns, all in the space of a few wavelengths. As far as can be seen in the classical approximation where there is no diffraction, the energy E (proportional to a^4) is constant while the phase P is discontinuous. Accordingly, this can be called the description of a phase-shock if the term plane chimera seems too extravagant.

A simpler, and equally interesting phase-shock is described by the case where $\ell=0$. Then $P_x=0$,

$$(3.5) \qquad \frac{1}{2} a_x^2 + V(a) = \varepsilon \ , \quad V(a) = \frac{1}{2}(\omega^2-k^2)a^2 - \frac{1}{4} a^4 \ ,$$

and the qualitative structure of the solutions can be inferred from figure 2.

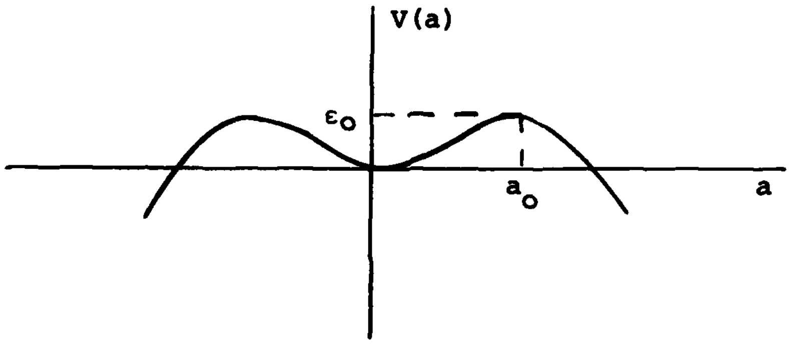


Figure 2a. V(a).

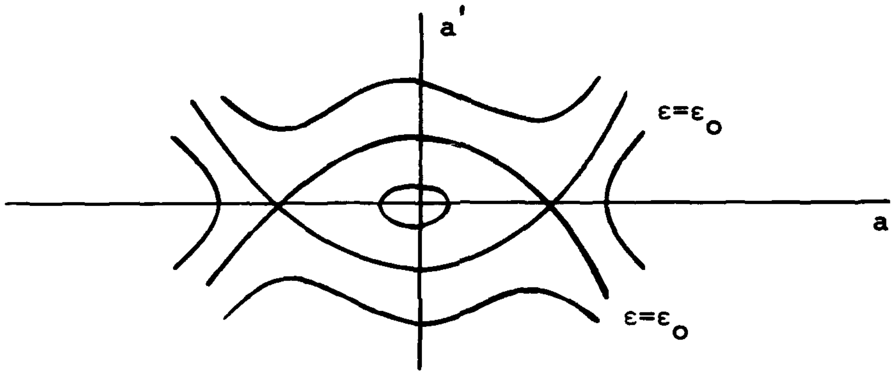


Figure 2b. The phase plane.

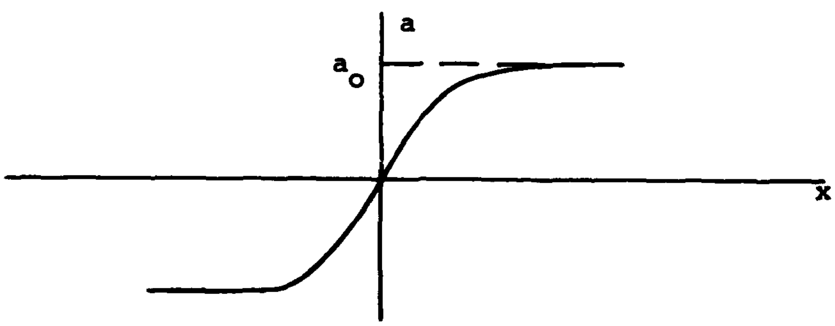


Figure 2c. $(\varepsilon=\varepsilon_o)$ A solution.

Again there are periodic plane chimeras when ε lies between zero and ε_o, and again the most interesting one is the one depicted in figure 3c where $\varepsilon = \varepsilon_o$ and the solution connects both hyperbolic points in the phase plane. In the classical approximation the energy E (still proportional to a^4) is constant and the magnitude of the discontinuity of the phase is π .

Cylindrical chimeras.

In the cylindrical coordinates (ρ,ϕ,z) equations (3.1) and (3.2) imply

$$(3.6) \qquad a^2 P_\rho = \frac{\ell}{\rho} , \quad a_{\rho\rho} + \frac{1}{\rho} a_\rho + (\omega^2 - k^2 - P_\rho^2 - a^2) a = 0$$

in the case where a and P_ρ are functions of ρ alone, P_ϕ is zero, and P_z is the constant k. If a and P_ρ are to remain bounded as $\rho \to 0$, one of the two has to vanish - the choice $P_\rho = 0$ is the only one that leads anywhere. The second equation (3.6) doesn't have an energy integral, but the qualitative features of its solutions can be illustrated by defining

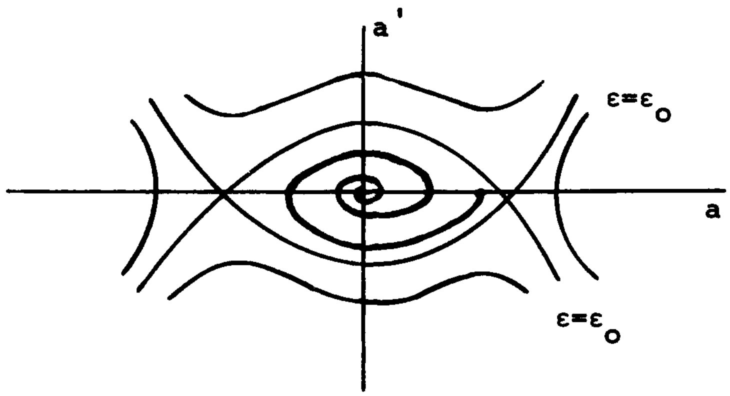
$$(3.7) \qquad \varepsilon(\rho) \equiv \frac{1}{2} a_\rho^2 + V(a) , \quad V(a) = \frac{1}{2}(\omega^2 - k^2) a^2 - \frac{1}{4} a^4 .$$

(A Liapunov function - same argot, some of the same applied mathematicians.) Then by a direct evaluation and the use of the differential equation, it follows that

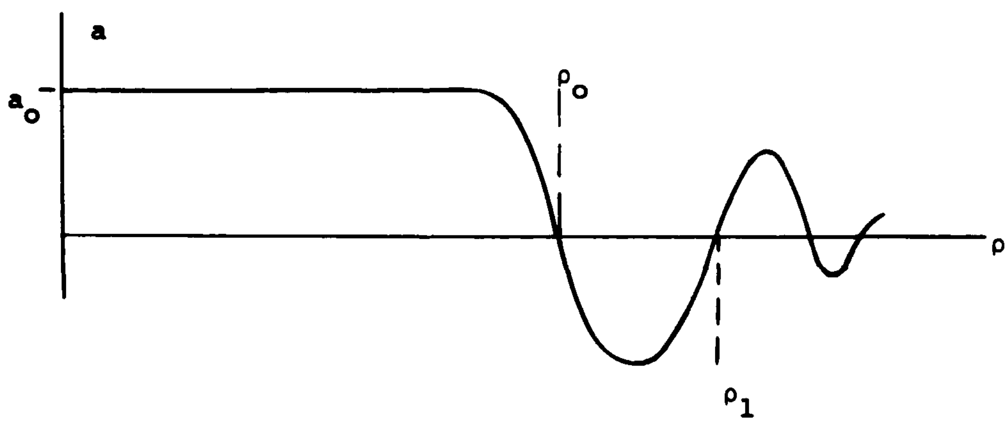
$$(3.8) \qquad \varepsilon'(\rho) = -\frac{1}{\rho} a_\rho^2 \leq 0$$

Thus the traces of the solutions of (3.7) and (3.8) in the phase plane always cross the lines ε = constant in such manner that ε decreases as ρ increases, making a third-order contact at points where a_ρ is zero, as depicted in figure 3.

<u>Figure 3a.</u> Same as figure 2a.



<u>Figure 3b</u>. The phase plane.



<u>Figure 3c</u>. A solution.

The solution depicted in figure 3c exhibits an interesting, absolutely nonlinear phenomenon. By choosing the initial point where ρ is zero to be one where a_ρ is zero and a is almost but not quite at the hyperbolic point a_o where $V'(a) = 0$, the solution can be made to hover in the neighborhood of its initial value out to an arbitrarily large radius before it finally spirals into the value a=0 as ρ approaches infinity. Ultimately the balance between diffraction and nonlinear dispersion is overcome by geometrical attenuation, and in two or more space dimensions the chimeras are fundamentally different from the one-dimensional varieties - they are localized.

A number of asymptotic properties of the solutions can be found: As $\rho \to \infty$, for example, they behave like Bessel functions of order zero, i.e. $a \sim a_1(\kappa\rho)^{-\frac{1}{2}}\cos(\kappa\rho+\phi)$ where $\kappa = (\omega^2-k^2)^{\frac{1}{2}}$ and a_1 and ϕ are constants that can't easily be evaluated. Another one that is fairly easy to find out is that as the radius of the main beam increases, the widths of the first few diffraction rings decrease relative to it according to the asymptotics, $(\rho_1-\rho_o)/\rho_o = O((\ell n\ \kappa\rho_o)/\kappa\rho_o)$ as $\kappa\rho_o \to \infty$. Thus we have a cylindrical beam of whatever it is that is radiated by (1.2), of arbitrary radius, of unchanging shape, surrounded by its diffraction rings, and propagated forever - a self-focussed beam.

Spherical chimeras.

In the spherical coordinates (r,θ,ϕ) the only spherically symmetric solution of (3.1) that makes sense as $r \to 0$ is the one where $P_r = P_\theta = P_\phi = 0$, and thus (3.2) becomes

$$(3.9) \qquad \varepsilon(r) \equiv \frac{1}{2} a_r^2 + \frac{1}{2} \omega^2 a^2 - \frac{1}{4} a^4$$

$$\varepsilon'(r) = -2a_r^2/r \ .$$

Since the points in the phase plane where $a_r=0$, $a=\pm\omega$ and $\varepsilon=\omega^4/4$ are hyperbolic points, there are spherical chimeras within which a is very nearly equal to $\pm\omega$ out to an arbitrary radius r_o. Beyond this there are diffraction rings separated by spherical phase shocks (where a^4 is zero) - the first few are strong ones with widths of $O(\ell n\ \omega r_o)$ when $\omega r_o >> 1$; in the far field they are weak in accord with the asymptotic behavior, $a \sim a_1 \cos(\omega r+\phi)/\omega r$ as $r \to \infty$, and their widths are of $O(\pi/\omega)$. Depending on what it is that (1.2) was meant to describe, we may wish to compare the properties of spherical chimeras with those of the transparent pulses of laser optics, or we may prefer to think of them as particles. A chimera moving at the velocity V along the x^1-axis can be described in terms of the Lorentz transformation

$$(3.10) \quad \bar{x}^1 = \frac{x^1-Vx^o}{(1-V^2)^{\frac{1}{2}}} , \quad \bar{x}^o = \frac{x^o-Vx^1}{(1-V^2)^{\frac{1}{2}}} , \quad \bar{x}^2 = x^2 , \quad \bar{x}^3 = x^3$$

$$\bar{k}^1 = 0 = \frac{k^1-Vk^o}{(1-V^2)^{\frac{1}{2}}} , \ \bar{k}^o = \omega = \frac{k^o-Vk^1}{(1-V^2)^{\frac{1}{2}}} , \quad \bar{k}^2=k^2=\bar{k}^3=k^3=0 \ .$$

where $(\bar{x}^k)$ is the rest frame, (x^k) is the laboratory. When the solution is spherically symmetric in the rest frame, the result

$$(3.11) \quad \phi(\underline{x},t) \sim A([\frac{(x-Vt)^2}{1-V^2} + y^2+z^2]^{\frac{1}{2}})F_1(\frac{\omega}{(1-V^2)^{\frac{1}{2}}}(t-Vx))$$

describes, as expected, a spheriodal wave packet moving with the group velocity V, made up of not quite sinusoidal waves that move through it with the phase velocity 1/V.

<u>Other creatures of the neoclassical bestiary</u>.

With the alternate measure of simplicity suggested in §1, (1.3) is bewilderingly simple. Only the most obvious chimeras have been described here - there are many others just waiting to be tamed. Some chimeras lurk in other coordinate systems and others masquerade as similarity solutions, for example, and there are numerical chimeras. Equation (1.3), if it is accepted as describing qualitative features of solutions of (1.2), is suitable for describing them in the large. In any region of space and time where a and $P_{,i}$ vary relatively slowly a numerical integration of (1.3) can be allowed to march in steps that span many wavelengths of the local wave. In an attempt at the numerical integration of (1.2) under similar circumstances, the machine prints gibberish that depends as much on the properties of round-off and truncation errors as it does on the properties of (1.2).

4. Breathing flame.

Whatever (1.2) propagates, its neoclassical approximation has a conservation law, the transport equation

$$g^{ij}(a^2 P_{,i})_{,j} = 0 , \tag{4.1}$$

that simply wasn't there to begin with. It appears because (2.6) depends on $P_{,i}$ but not on P, and evidently one like it will be found in the neoclassical approximation of any nonlinear (or linear) wave equation that is derived from a stationary principle. Actually its appearance can be traced all the way back to the representation (2.5) - the derivation of the bifurcation condition (2.10b) didn't really depend on the representation (2.7) at all (just substitute f for F). In terms of the mechanical analog of the local nonlinear oscillation (4.1) is the generalization for partial differential equations of what appears as adiabatic invariance for ordinary differential equations. There is a temptation to call it conservation of mass, but that would be a guess, and quite possibly a wrong one. So far the only solution we have that looks like a particle looks like a free particle, and not much can be said about its mass until interactions have been dealt with. Furthermore, if we rewrite (4.1) as

$$(\rho_o u^i)_{,i} = 0 \tag{4.2}$$

where

$$u^i \equiv g^{ij} P_{,j} / (g^{k\ell} P_{,k} P_{,\ell})^{\frac{1}{2}} \tag{4.3}$$

is a four-velocity field with the property $u^i u_i = 1$ and

$$\rho_o \equiv (g^{k\ell} P_{,k} P_{,\ell})^{\frac{1}{2}} a^2 \tag{4.4}$$

is a scalar field we choose to call rest density, then the rest mass of spherical chimera

$$m_o = 4\pi \int_o^\infty \rho_o r^2 dr = 4\pi\omega \int_o^\infty a^2 r^2 dr \tag{4.5}$$

diverges on account of the asymptotic behavior

$$a \sim a_1 \cos(\omega r + \phi)/\omega r \quad \text{as} \quad r \to \infty \, . \tag{4.6}$$

The divergence of m_o may or may not be disappointing, but it is not surprising in view of the fact that there is a relativistic quantum field theory to be found among the approximations of the style we have been discussing.

In the previous section we found, in the spherical chimera, a solution of the neoclassical approximation that has properties somewhat like those of a relativistic free particle; and at the same time it is definitely a wave-packet. It would have been a surprise if the Schrodinger equation couldn't be found here. It can, but to see it we shall have to redo the approximate treatment of (1.2), practically from the beginning, taking the similarity of $F_1(\theta)$ to sin θ more seriously than was ever intended. The only use of that similarity, it may be recalled, was for the estimation of the values of the coefficients α and β in the diffraction-dispersion equation (2.30), after which it was

asserted that (1.3) is an adequate approximation. (In other problems governed by (2.1) the wave form of a plane wave needn't be well approximated by a sine wave.) Now (1.3) was also obtained from (1.2) by the use of the one-term Fourier representation, $F(\theta;A) = A(x^k)\sin\theta$, and we know it is not a bad approximation. Given that, we can complicate the problem by working with the complex representation

$$(4.7) \quad F(\theta;\psi) = \tfrac{1}{2}(\psi(x^k)e^{i\theta}+\psi^*(x^k)e^{-i\theta})$$

instead. Then

$$(4.8) \quad \mathcal{L} = \tfrac{1}{8}\, g^{ij}[P_{,i}P_{,j}\psi\psi^*+\psi_{,i}\psi^*_{,j}+i(P_{,i}\psi\psi^*_{,j}-P_{,j}\psi_{,i}\psi^*)]-\tfrac{3}{32}(\psi\psi^*)^2$$

implies ψ^* is the complex conjugate of ψ,

$$(4.9) \quad g^{ij}(\psi\psi^*P_{,i}+\tfrac{1}{2}\, i(\psi\psi^*_{,i}-\psi_{,i}\psi^*))_{,j}=0$$

and

$$(4.10) \quad g^{ij}(\psi_{,ij}+i(P_{,ij}\psi+2P_{,i}\psi_{,j})) = (g^{ij}P_{,i}P_{,j} - \tfrac{3}{2}\,\psi\psi^*)\psi$$

- the quantum mechanical approximation. In the rest frame of any solution that has a rest frame where $(P_{,i}) = (\omega,\underline{0})$, $P_{,ij} = 0$ and $\psi_{,0} = 0$, it collapses to

$$(4.11) \quad \nabla^2\psi + (\omega^2 - \tfrac{3}{2}\,|\psi|^2)\psi = 0 ,$$

as expected for a free particle.

If it had happened that the quantum mechanical approximation was nothing more than an unnecessary complication of the neoclassical approximation, applicable in cases where the local wave form is well approximated by a sine wave - that would have been a most unkind cut, but that is not the case. Another way to look at the complex representation (4.7) is to write it as

$$(4.12) \qquad F(\theta;\psi) = F(\theta+\chi;A) = A(x^k)\sin(\theta+\chi(x^k))$$

where

$$(4.13) \qquad A = |\psi\psi^*|^{\frac{1}{2}} \quad \text{and} \quad \tan\chi = -\,\mathrm{Re}(\psi)/\mathrm{Im}(\psi)$$

and the branch of arctangent to be used for the evaluation of χ changes with the addition of $\pm\pi$ to χ whenever the sign of A, as determined by the one-term Fourier approximation, changes; and still another way to write it is to let $A = \pm|\psi\psi^*|^{\frac{1}{2}}$ and fix the branch of arctangent once and for all. The representation (4.12) leads right back to the one-term Fourier approximation, except that $P(x^k)$ is to be replaced by $P(x^k)+\chi(x^k)$, thus indicating that $P(x^k)$ can be chosen arbitrarily. This can also be seen by showing directly that (4.9) follows from (4.10) and its complex conjugate for any $P(x^k)$. In particular we may set $(P_{.i}) = (\omega,\underline{0})$ in any convenient frame to define a fixed-frame quantum mechanical approximation

$$(4.14) \qquad \psi_{tt}-\nabla^2\psi + 2i\omega\psi_t = (\omega^2 - \tfrac{3}{2}|\psi|^2)\psi.$$

All the information contained in the two real equations of the one-term Fourier approximation is present in the single complex

equation (4.14). This may well lead to a simplification, depending on the type of problem that is to be considered. In the discussion of solutions that are almost steady in some frame (the rest frame), for example, we may wish to consider the case where $|\psi_{tt}| \ll |\omega\psi_t|$ and

$$(4.15) \qquad \nabla^2\psi + (\omega^2 - \tfrac{3}{2}|\psi|^2)\psi \sim 2i\omega\psi_t \; .$$

Equation (4.15) is not covariant in general; the inequality on which the approximation is based can be expected only to hold in some rest frame and in frames that move with uniform velocities much less than one relative to it. In other words, it is the non-relativistic approximation of (4.14). Equation (4.14), on the other hand, can be called covariant or not - the choice is still open. If we choose to call ψ a complex scalar field, then we have to replace $\omega\psi_t$ by $\omega^i\psi_{,i}$ in it to arrive at a covariant formulation. It has been argued, however, that (4.14) can be written as is in any frame, and if we choose to do so, then ψ is not a scalar field, but has the more complicated transformation law implied by equations (4.12) and (4.13) when A and $\omega x^o+\chi$ are scalar fields.

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13. ABSTRACT

The propagation of nonlinear wave packets and modulated beams, and other features of nonlinear wave propagation can be described in terms of several approximate, heuristic theories. The neoclassical approximation, to be presented here, is representative of a style of approximation that includes a number of the approximations currently in use as special cases or as limiting cases. It is, in a manner of speaking, the best approximation of its kind in the sense that it coincides with the nonlinear wave equation from which it is derived in more limiting cases than any of the others, and though it too is heuristic, it is rational, being the first of a sequence of approximations that converges to a solution if it converges. Chimeras are the solutions of the equations of the neoclassical approximation; there are a number of exact ones, to be exhibited here, that described phase-shocks, self-focussed beams, and localized wave packets that travel, without change of shape, at an arbitrary uniform velocity less than one.

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